

The number of cliques in graphs of given order and size

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February 19, 2008

Abstract

Let $k_r(n, m)$ denote the minimum number of r -cliques in graphs with n vertices and m edges. For $r = 3, 4$ we give a lower bound on $k_r(n, m)$ that approximates $k_r(n, m)$ with an error smaller than $n^r / (n^2 - 2m)$.

The solution is based on a constraint minimization of certain multilinear forms. Our proof combines a combinatorial strategy with extensive analytical arguments.

AMS classification:

Keywords: *number of cliques; multilinear forms; Turán graph.*

Introduction

Our graph-theoretic notation follows [3]; in particular, an r -clique is a complete subgraph on r vertices.

What is the minimum number $k_r(n, m)$ of r -cliques in graphs with n vertices and m edges? This problem originated with the famous graph-theoretical theorem of Turán more than sixty years ago, but despite numerous attempts, never got a satisfactory solution, see [2], [4], [5], [6], [7], and [9] for some highlights of its long history. Most recently, the problem was discussed in detail in [1].

The best result so far is due to Razborov [9]. Applying tools developed in [8], he achieved a remarkable progress for $r = 3$. But this method failed for $r > 3$, and Razborov challenged the mathematical community to extend his result.

The aim of this paper is to answer this challenge. We introduce a class of multilinear forms and find their minima subject to certain constraints. As a consequence, for $r = 3, 4$ we obtain a lower bound on $k_r(n, m)$, approximating $k_r(n, m)$ with an error smaller than $n^r / (n^2 - 2m)$.

In our proof, a combinatorial main strategy cooperates with analytical arguments using Taylor's expansion, Lagrange's multipliers, compactness, continuity, and connectedness. We believe that such cooperation can be developed further and applied to other problems in extremal combinatorics.

It seems likely that these methods will enable the solution of the problem for $r > 4$ as well. With this idea in mind we present all results as general as possible.

1 Main results

Suppose $1 \leq r \leq n$, let $[n] = \{1, \dots, n\}$, and write $\binom{[n]}{r}$ for the set of r -subsets of $[n]$. For a symmetric $n \times n$ matrix $A = (a_{ij})$ and a vector $\mathbf{x} = (x_1, \dots, x_n)$, set

$$L_r(A, \mathbf{x}) = \sum_{X \in \binom{[n]}{r}} \prod_{i, j \in X, i < j} a_{ij} \prod_{i \in X} x_i. \quad (1)$$

Define the set $\mathcal{A}(n)$ of symmetric $n \times n$ matrices $A = (a_{ij})$ by

$$\mathcal{A}(n) = \{A : a_{ii} = 0 \text{ and } 0 \leq a_{ij} = a_{ji} \leq 1 \text{ for all } i, j \in [n]\}.$$

Our main goal is to find $\min L_r(A, \mathbf{x})$ subject to the constraints

$$A \in \mathcal{A}(n), \quad \mathbf{x} \geq 0, \quad L_1(A, \mathbf{x}) = b, \quad \text{and} \quad L_2(A, \mathbf{x}) = c,$$

where b and c are fixed positive numbers. Since every $L_s(A, \mathbf{x})$ is homogenous of first degree in each x_i , for simplicity we assume that $b = 1$ and study

$$\min \{L_r(A, \mathbf{x}) : (A, \mathbf{x}) \in \mathcal{S}_n(c)\}, \quad (2)$$

where $\mathcal{S}_n(c)$ is the set of pairs (A, \mathbf{x}) defined as

$$\mathcal{S}_n(c) = \{(A, \mathbf{x}) : A \in \mathcal{A}(n), \mathbf{x} \geq 0, L_1(A, \mathbf{x}) = 1, \text{ and } L_2(A, \mathbf{x}) = c\}.$$

Note that $\mathcal{S}_n(c)$ is compact since the functions $L_s(A, \mathbf{x})$ are continuous; hence (2) is defined whenever $\mathcal{S}_n(c)$ is nonempty. The following proposition, proved in 2.1, describes when $\mathcal{S}_n(c) \neq \emptyset$.

Proposition 1.1 *$\mathcal{S}_n(c)$ is nonempty if and only if $c < 1/2$ and $n \geq \lceil 1/(1-2c) \rceil$.*

Hereafter we assume that $0 < c < 1/2$ and set $\xi(c) = \lceil 1/(1-2c) \rceil$.

To find (2), we solve a seemingly more general problem: for all $c \in (0, 1/2)$, $n \geq \xi(c)$, and $3 \leq r \leq n$, find

$$\varphi_r(n, c) = \min \{L_r(A, \mathbf{x}) : r \leq k \leq n, (A, \mathbf{x}) \in \mathcal{S}_k(c)\}.$$

We obtain the solution of (2) by showing that, in fact, $\varphi_r(n, c)$ is independent of n .

To state $\varphi_r(n, c)$ precisely, we need some preparation. Set $s = \xi(c)$ and note that the system

$$\binom{s-1}{2} x^2 + (s-1)xy = c, \quad (3)$$

$$\begin{aligned} (s-1)x + y &= 1, \\ x &\geq y \end{aligned} \quad (4)$$

has a unique solution

$$x = \frac{1}{s} + \frac{1}{s} \sqrt{1 - \frac{2s}{s-1}c}, \quad y = \frac{1}{s} - \frac{s-1}{s} \sqrt{1 - \frac{2s}{s-1}c}. \quad (5)$$

Write \mathbf{x}_c for the s -vector (x, \dots, x, y) and let $A_s \in \mathcal{A}(s)$ be the matrix with all off-diagonal entries equal to 1. Note that equations (3) and (4) give $(A_s, \mathbf{x}_c) \in \mathcal{S}_s(c)$.

Setting $\varphi_r(c) = L_r(A_s, \mathbf{x}_c)$, we arrive at the main result in this section.

Theorem 1.2 *If $c \in (0, 1/2)$, $r \in \{3, 4\}$, and $r \leq \xi(c) \leq n$, then $\varphi_r(n, c) = \varphi_r(c)$.*

Note first that the premise $r \leq \xi(c)$ is not restrictive, for, $\varphi_r(n, c) = 0$ whenever $r > \xi(c)$. Indeed, assume that $r > \xi(c)$ and write \mathbf{y} for the r -vector $(x, \dots, x, y, 0, \dots, 0)$ whose last $r - s$ entries are zero. Writing B for the $r \times r$ matrix with A_s as a principal submatrix in the first s rows and with all other entries being zero, we see that $(B, \mathbf{y}) \in \mathcal{S}_r(c)$ and $L_r(B, \mathbf{y}) = 0$; hence $\varphi_r(n, c) = 0$, as claimed.

Next, note an explicit form of $\varphi_r(c)$:

$$\begin{aligned} \varphi_r(c) &= \binom{s-1}{r} x^r + \binom{s-1}{r-1} x^{r-1} y \\ &= \binom{s}{r} \frac{1}{s^r} \left(1 - (r-1) \sqrt{1 - \frac{2s}{s-1} c} \right) \left(1 + \sqrt{1 - \frac{2s}{s-1} c} \right)^{r-1}. \end{aligned}$$

Since $\varphi_r(c)$ is defined via the discontinuous step function $\xi(c)$, the following properties of $\varphi_r(c)$ are worth stating:

- $\varphi_r(c)$ is continuous for $c \in (0, 1/2)$;
- $\varphi_r(c) = 0$ for $c \in (0, 1/4]$ and is increasing for $c \in (1/4, 1/2)$;
- $\varphi_r(c)$ is differentiable and concave in any interval $((s-1)/2s, s/2(s+1))$.

1.1 The number of cliques

Write $k_r(G)$ for the number of r -cliques of a graph G and let us outline the connection of Theorem 1.2 to $k_r(G)$. Let

$$k_r(n, m) = \min \{k_r(G) : G \text{ has } n \text{ vertices and } m \text{ edges}\},$$

and suppose that $k_r(n, m)$ is attained on a graph G with adjacency matrix $A = (a_{ij})$. Clearly, for every $X \in \binom{[n]}{r}$,

$$\prod_{i,j \in X, i < j} a_{ij} = \begin{cases} 1, & \text{if } X \text{ induces an } r\text{-clique in } G, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, letting $\mathbf{x} = (1/n, \dots, 1/n)$, we see that

$$L_1(A, \mathbf{x}) = 1, \quad L_2(A, \mathbf{x}) = m/n^2, \quad \text{and} \quad L_r(A, \mathbf{x}) = k_r(G)/n^r;$$

thus Theorem 1.2 gives

$$k_r(n, m) \geq \varphi_r(n, m/n^2) n^r = \varphi_r(m/n^2) n^r.$$

Setting $s = \xi(m/n^2) = \lceil 1/(1 - 2m/n^2) \rceil$, we obtain an explicit form of this inequality

$$k_r(n, m) \geq \binom{s}{r} \frac{1}{s^r} \left(n - (r-1) \sqrt{n^2 - \frac{2sm}{s-1}} \right) \left(n + \sqrt{n^2 - \frac{2sm}{s-1}} \right)^{r-1}. \quad (6)$$

Inequality (6) turns out to be rather tight, as stated below and proved in Section 3.

Theorem 1.3

$$k_r(n, m) < \varphi_r\left(\frac{m}{n^2}\right) n^r + \frac{n^r}{n^2 - 2m}.$$

Note, in particular, that if $m < (1/2 - \varepsilon) n^2$, then

$$k_r(n, m) < \varphi_r(m/n^2) n^r + n^{r-2}/2\varepsilon,$$

so the order of the error is lower than expected.

Known previous results

For $n^2/4 \leq m \leq n^2/3$ inequality (6) was first proved by Fisher [6]. He showed that

$$k_3(n, m) \geq \frac{9nm - 2n^3 - 2(n^2 - 3m)^{3/2}}{27} = \varphi_3(m/n^2) n^3,$$

but did not discuss how close the two sides of this inequality are.

Recently Razborov [9] showed that for every fixed $c \in (0, 1/2)$,

$$k_3(n, \lceil cn^2 \rceil) = \varphi_3(c) n^3 + o(n^3).$$

Unfortunately, his approach, based on [8], provides no clues whatsoever how large the $o(n^3)$ term is; in particular, in his approach this term is not uniformly bounded when c approaches $1/2$. In [9] Razborov challenged the mathematical community to prove that $k_r(n, \lceil cn^2 \rceil) = \varphi_r(c) n^r + o(n^r)$ for $r > 3$. Our Theorem 1.2 proves this equality for $r = 4$.

2 Proof of Theorem 1.2

The following simple lemma will be used in the proof of Theorem 1.2.

Lemma 2.1 *Let $0 \leq c \leq a$ and $0 \leq d \leq b$. If $0 \leq x \leq \min(a, b)$ and $0 \leq y \leq \min(c, d)$, then*

$$(a - c)(b - d) + x(c + d) + y(a + b) - (x + y)^2 \geq 0$$

Proof Set $P = x(c + d) + y(a + b) - (x + y)^2$. Since $(a - c)(b - d) \geq 0$, we may and shall suppose that $P < 0$. By symmetry, we also suppose that $a \geq b$. If $x + y \leq b$, by $c + d \leq a + b$ we have

$$P \geq (x + y)(c + d) + y(a + b - c - d) - (x + y)^2 \geq (x + y)(c + d) - (x + y)^2;$$

hence, $P < 0$ implies that $b > c + d$ and $P \geq b(c + d) - b^2$. Now the proof is completed by

$$(a - c)(b - d) + b(c + d) - b^2 = (a - b)(b - d) + cd > 0.$$

If $x + y > b$, by $c + d \leq a + b$, we have

$$P \geq b(c + d) + y(a + b) - (b + y)^2 = b(c + d) + y(a - b) - b^2 - y^2;$$

hence, $P < 0$ implies that $\min(c, d) > a - b$ and

$$P \geq b(c + d) + \min(c, d)(a - b) - b^2 - (\min(c, d))^2.$$

If $d \geq c$, we get

$$\begin{aligned} (a - c)(b - d) + P &\geq (a - c)(b - d) - b(b - d) + c(a - c) \\ &\geq (a - c)(b - d) - b(b - d) + c(b - d) = (a - b)(b - d) \geq 0. \end{aligned}$$

If $c \geq d$, we get

$$\begin{aligned} (a - c)(b - d) + P &\geq (a - c)(b - d) + b(c + d) + d(a - b) - b^2 - d^2 \\ &= a(a - b) + c(c - d) \geq 0, \end{aligned}$$

completing the proof of Lemma 2.1. \square

Next we show that $\varphi_r(n, c)$ increases in c whenever $\varphi_r(n, c) > 0$.

Proposition 2.2 *Let $c \in (0, 1/2)$ and $3 \leq r \leq \xi(c) \leq n$. If $\varphi_r(n, c) > 0$ and $0 < c_0 < c$, then $\varphi_r(n, c) > \varphi_r(n, c_0)$.*

Proof Suppose that

$$\xi(c) \leq k \leq n, \quad (A, \mathbf{x}) \in \mathcal{S}_k(c), \quad \text{and} \quad \varphi_r(n, c) = L_r(A, \mathbf{x}).$$

Setting $\alpha = c_0/c$, we see that $\alpha A \in \mathcal{A}(k)$ and

$$L_2(\alpha A, \mathbf{x}) = \alpha L_r(A, \mathbf{x}) = c_0;$$

thus $(\alpha A, \mathbf{x}) \in \mathcal{S}_k(c_0)$. Hence we obtain

$$\varphi_r(n, c) = L_r(A, \mathbf{x}) = \alpha^{-\binom{r}{2}} L_r(\alpha A, \mathbf{x}) > L_r(\alpha A, \mathbf{x}) \geq \varphi_r(n, c_0),$$

completing the proof of Proposition 2.2. \square

Proof of Theorem 1.2

Let us first define a set of n -vectors $\mathcal{X}(n)$ by

$$\mathcal{X}(n) = \{(x_1, \dots, x_n) : x_1 + \dots + x_n = 1 \text{ and } x_i \geq 0, 1 \leq i \leq n\}.$$

Now the conditions $\mathbf{x} \in \mathcal{X}(n)$ is equivalent to $\mathbf{x} \geq 0$ and $L_1(A, \mathbf{x}) = 1$.

Assume for a contradiction that the theorem fails: let

$$c \in (0, 1/2), \quad 3 \leq r \leq \xi(c) \leq n, \quad A = (a_{ij}), \quad \mathbf{x} = (x_1, \dots, x_n), \quad \text{and} \quad (A, \mathbf{x}) \in \mathcal{S}_n(c) \quad (7)$$

be such that

$$\varphi_r(n, c) = L_r(A, \mathbf{x}) < \varphi_r(c). \quad (8)$$

Assume that n is the minimum integer with this property for all $c \in (0, 1/2)$, and that, among all pairs $(A, \mathbf{x}) \in \mathcal{S}_n(c)$, A has the maximum number of zero entries. Hereafter we shall refer to this assumption as the “main assumption”. The most important consequence of the main assumption is the following

Claim 2.3 *If $(A, \mathbf{y}) \in \mathcal{S}_n(c)$ and $\varphi_r(n, c) = L_r(A, \mathbf{y})$, then \mathbf{y} has no zero entries.* \square

Next we introduce some notation and conventions to simplify the presentation. For short, for every $i, j, \dots, k \in [n]$, set

$$C_i = \frac{\partial L_2(A, \mathbf{x})}{\partial x_i}, \quad C_{ij} = \frac{\partial L_2(A, \mathbf{x})}{\partial x_i \partial x_j}, \quad D_{ij\dots k} = \frac{\partial L_r(A, \mathbf{x})}{\partial x_i \partial x_j \cdots \partial x_k},$$

and note that

$$C_{ij} = a_{ij}, \quad \text{and} \quad \frac{\partial L_r(A, \mathbf{x})}{\partial a_{ij}} a_{ij} = D_{ij} x_i x_j. \quad (9)$$

Letting $\mathbf{y} = (x_1 + \Delta_1, \dots, x_n + \Delta_n)$, Taylor's formula gives

$$L_2(A, \mathbf{y}) - L_2(A, \mathbf{x}) = \sum_{i=1}^n C_i \Delta_i + \sum_{1 \leq i < j \leq n} C_{ij} \Delta_i \Delta_j \quad (10)$$

and

$$L_r(A, \mathbf{y}) - L_r(A, \mathbf{x}) = \sum_{s=1}^r \sum_{1 \leq i_1 < \dots < i_s \leq n} D_{i_1 \dots i_s} \Delta_{i_1} \cdots \Delta_{i_s}. \quad (11)$$

We shall use extensively Lagrange multipliers. Since $\mathbf{x} > 0$, by Lagrange's method, there exist λ and μ such that

$$D_i = \lambda C_i + \mu \quad (12)$$

for all $i \in [n]$. Likewise, if $0 < a_{ij} < 1$, we have

$$\frac{\partial L_r(A, \mathbf{x})}{\partial a_{ij}} = \lambda \frac{\partial L_2(A, \mathbf{x})}{\partial a_{ij}} = \lambda x_i x_j,$$

and so, in view of (9),

$$D_{ij} = \lambda a_{ij} \quad \text{whenever} \quad 0 < a_{ij} < 1. \quad (13)$$

The rest of the proof is presented in a sequence of formal claims. First we show that $\varphi_r(n, c)$ is attained on a $(0, 1)$ -matrix A .

Claim 2.4 *Let $(A, \mathbf{x}) \in \mathcal{S}_n(c)$ satisfy (7) and (8), and suppose that A has the smallest number of entries a_{ij} such that $0 < a_{ij} < 1$. Then A is a $(0, 1)$ -matrix.*

Proof Assume for a contradiction that $i, j \in [n]$ and $0 < a_{ij} < 1$. By symmetry we suppose that $C_i \geq C_j$. Let

$$f(\alpha) = \frac{a_{ij}\alpha^2 - (C_i - C_j)\alpha}{(x_i + \alpha)(x_j - \alpha)}, \quad (14)$$

and suppose that α satisfies

$$0 < \alpha < x_j \quad \text{and} \quad 0 \leq a_{ij} + f(\alpha) \leq 1. \quad (15)$$

Let $\mathbf{y}_\alpha = (x_1 + \Delta_1, \dots, x_n + \Delta_n)$, where

$$\Delta_i = \alpha, \quad \Delta_j = -\alpha, \quad \text{and} \quad \Delta_l = 0 \text{ for } l \in [n] \setminus \{i, j\}, \quad (16)$$

and define the $n \times n$ matrix $B_\alpha = (b_{ij})$ by

$$b_{ij} = b_{ji} = a_{ij} + f(\alpha) \quad \text{and} \quad b_{pq} = a_{pq} \text{ for } \{p, q\} \neq \{i, j\}. \quad (17)$$

Note that $B_\alpha \in \mathcal{A}(n)$, $\mathbf{y}_\alpha \in \mathcal{X}(n)$, and

$$L_2(B_\alpha, \mathbf{y}_\alpha) - L_2(A, \mathbf{y}_\alpha) = f(\alpha) \frac{\partial L_2(A, \mathbf{y}_\alpha)}{\partial a_{ij}} = f(\alpha) (x_i + \alpha) (x_j - \alpha).$$

Hence, Taylor's expansion (10) and equation (14) give

$$\begin{aligned} L_2(B_\alpha, \mathbf{y}_\alpha) - L_2(A, \mathbf{x}) &= L_2(A, \mathbf{y}_\alpha) - L_2(A, \mathbf{x}) + f(\alpha) (x_i + \alpha) (x_j - \alpha) \\ &= (C_i - C_j) \alpha - a_{ij} \alpha^2 + f(\alpha) (x_i + \alpha) (x_j - \alpha) = 0; \end{aligned}$$

thus $(B_\alpha, \mathbf{y}_\alpha) \in \mathcal{S}_n(c)$.

Note also that, in view of (9),

$$L_r(B_\alpha, \mathbf{y}_\alpha) - L_r(A, \mathbf{y}_\alpha) = \frac{\partial L_r(A, \mathbf{y}_\alpha)}{\partial a_{ij}} f(\alpha) = f(\alpha) y_i y_j \frac{D_{ij}}{a_{ij}} = f(\alpha) (x_i + \alpha) (x_j - \alpha) \frac{D_{ij}}{a_{ij}}.$$

Hence Taylor's expansion (11), Lagrange's conditions (12) and (13), and equation (14) give

$$\begin{aligned} L_r(B_\alpha, \mathbf{y}_\alpha) - L_r(A, \mathbf{x}) &= L_r(A, \mathbf{y}_\alpha) - L_r(A, \mathbf{x}) + f(\alpha) (x_i + \alpha) (x_j - \alpha) \frac{D_{ij}}{a_{ij}} \\ &= (D_i - D_j) \alpha - D_{ij} \alpha^2 + f(\alpha) (x_i + \alpha) (x_j - \alpha) \frac{D_{ij}}{a_{ij}} \\ &= \lambda (C_i - C_j) \alpha - D_{ij} \alpha^2 + f(\alpha) (x_i + \alpha) (x_j - \alpha) \frac{D_{ij}}{a_{ij}} \\ &= \frac{D_{ij}}{a_{ij}} (C_i - C_j) \alpha - D_{ij} \alpha^2 + f(\alpha) (x_i + \alpha) (x_j - \alpha) \frac{D_{ij}}{a_{ij}} \\ &= \frac{D_{ij}}{a_{ij}} ((C_i - C_j) \alpha - a_{ij} \alpha^2 + a_{ij} \alpha^2 - (C_i - C_j) \alpha) = 0. \end{aligned}$$

If there exists $\alpha \in (0, x_j)$ such that $a_{ij} + f(\alpha) = 0$ or $a_{ij} + f(\alpha) = 1$, we see that the matrix B_α has fewer entries belonging to $(0, 1)$ than A , contradicting the hypothesis and completing the proof. Assume therefore that $0 < a_{ij} + f(\alpha) < 1$ for all $\alpha \in (0, x_j)$. This condition implies that

$$a_{ij} x_j = C_i - C_j,$$

for, otherwise $\lim_{\alpha \rightarrow x_j} |f(\alpha)| = \infty$, and so, either $a_{ij} + f(\alpha) = 0$ or $a_{ij} + f(\alpha) = 1$ for some $\alpha \in (0, x_j)$.

Now, extending $f(\alpha)$ continuously for $\alpha = x_j$ by

$$f(x_j) = \lim_{\alpha \rightarrow x_j} f(\alpha) = \lim_{\alpha \rightarrow x_j} \frac{a_{ij}\alpha(\alpha - x_j)}{(x_i + \alpha)(x_j - \alpha)} = -\frac{a_{ij}x_j}{x_i + x_j},$$

and defining \mathbf{y}_{x_j} by (16) and B_{x_j} by (17), we obtain

$$L_r(B_{x_j}, \mathbf{y}_{x_j}) - \varphi_r(n, c) = L_r(B_{x_j}, \mathbf{y}_{x_j}) - L_r(A, \mathbf{x}) = 0.$$

contradicting Claim 2.3 since the j th entry of \mathbf{y}_{x_j} is zero. This completes the proof of Claim 2.4. \square

Since A is a $(0, 1)$ -matrix with a zero main diagonal, it is the adjacency matrix of some graph G with vertex set $[n]$. Write $E(G)$ for the edge set of G and let us restate the functions $L_r(A, \mathbf{x})$ in terms of G . We have

$$L_2(A, \mathbf{x}) = \sum_{ij \in E(G)} x_i x_j$$

and more generally,

$$L_r(A, \mathbf{x}) = \sum \{x_{i_1} \cdots x_{i_r} : \text{the set } \{i_1, \dots, i_r\} \text{ induces an } r\text{-clique in } G\}.$$

To finish the proof of Theorem 1.2 we show that G is a complete graph and $L_r(A, \mathbf{x}) = \varphi_r(c)$.

Proof that G is a complete graph

For convenience we first outline this part of the proof. Write \overline{G} for the complement of G and $E(\overline{G})$ for the edge set of \overline{G} . We assume that G is not complete and reach a contradiction by the following major steps:

- if $ij \in E(\overline{G})$, then $C_i \neq C_j$ - Claim 2.5;
- if $ij \in E(G)$, then $D_{ij} < \lambda$ - Claim 2.6;
- \overline{G} is triangle-free - Claim 2.7;
- \overline{G} is bipartite - Claims 2.8 and 2.9;
- G contains induced 4-cycles - Claim 2.10;
- G contains no induced 4-cycles - Claim 2.11.

Now the details.

Claim 2.5 *If $ij \in E(\overline{G})$, then $C_i \neq C_j$.*

Proof Assume that $ij \in E(\overline{G})$ and $C_i = C_j$. Let $\mathbf{y} = (x_1 + \Delta_1, \dots, x_n + \Delta_n)$, where

$$\Delta_i = -x_i, \quad \Delta_j = x_i, \quad \text{and} \quad \Delta_l = 0 \text{ for } l \in [n] \setminus \{i, j\}.$$

Clearly, $\mathbf{y} \in \mathcal{X}(n)$; Taylor's expansion (10) gives

$$L_2(A, \mathbf{y}) - L_2(A, \mathbf{x}) = C_j x_i - C_i x_i = 0;$$

thus, $(A, \mathbf{y}) \in \mathcal{S}_n(c)$. Taylor's expansion (11) and Lagrange's condition (12) give

$$L_r(A, \mathbf{y}) - L_r(A, \mathbf{x}) = D_j x_i - D_i x_i = \mu(x_i - x_i) + \lambda(C_j - C_i)x_i = 0,$$

contradicting Claim 2.3 as the i th entry of \mathbf{y} is zero. The proof of Claim 2.5 is completed. \square

Claim 2.6 *If $ij \in E(G)$, then $D_{ij} < \lambda$.*

Proof Assume that $ij \in E(G)$ and $D_{ij} \geq \lambda$. Select $pq \in E(\overline{G})$; by Claim 2.5 suppose that $C_p > C_q$. For every $\alpha \in (0, x_q)$, let $\mathbf{y}_\alpha = (y_1, \dots, y_n)$, where

$$y_p = x_p + \alpha, \quad y_q = x_q - \alpha, \quad \text{and} \quad y_l = x_l \text{ for all } l \in [n] \setminus \{p, q\}.$$

Let

$$f(\alpha) = \frac{(C_q - C_p)\alpha}{y_i y_j}. \quad (18)$$

and define the $n \times n$ matrix $B_\alpha = (b_{rs})$ by

$$b_{ij} = b_{ji} = 1 + f(\alpha), \quad \text{and} \quad b_{rs} = a_{rs} \text{ for } \{r, s\} \neq \{i, j\}.$$

For α sufficiently small, $-1 < f(\alpha) < 0$, and so $B_\alpha \in \mathcal{A}(n)$ and $\mathbf{y}_\alpha \in \mathcal{X}(n)$. Taylor's expansion (10) and equation (18) give

$$\begin{aligned} L_2(B_\alpha, \mathbf{y}_\alpha) - L_2(A, \mathbf{x}) &= L_2(B_\alpha, \mathbf{y}_\alpha) - L_2(A, \mathbf{y}_\alpha) + L_2(A, \mathbf{y}_\alpha) - L_2(A, \mathbf{x}) \\ &= f(\alpha) y_i y_j + \alpha(C_p - C_q) = 0; \end{aligned}$$

thus, $(B_\alpha, \mathbf{y}_\alpha) \in \mathcal{S}_n(c)$.

Taylor's expansion (11), Lagrange's condition (12), and equation (18) give

$$\begin{aligned} L_r(B_\alpha, \mathbf{y}_\alpha) - L_r(A, \mathbf{x}) &= L_r(B_\alpha, \mathbf{y}_\alpha) - L_r(A, \mathbf{y}_\alpha) + L_r(A, \mathbf{y}_\alpha) - L_r(A, \mathbf{x}) \\ &= D_p \alpha - D_q \alpha + D_{ij} f(\alpha) y_i y_j = \lambda(C_p - C_q) \alpha - D_{ij} (C_p - C_q) \alpha \\ &= \alpha(C_p - C_q)(\lambda - D_{ij}). \end{aligned}$$

Since $L_r(B_\alpha, \mathbf{y}_\alpha) \geq L_r(A, \mathbf{x})$, $\alpha(C_p - C_q) > 0$, and $D_{ij} \geq \lambda$, we see that $L_r(B_\alpha, \mathbf{y}_\alpha) = L_r(A, \mathbf{x})$.

If there exists $\alpha \in (0, x_q)$ such that $a_{ij} + f(\alpha) = 0$, then the $(0, 1)$ -matrix B_α has more zero entries than A , contradicting the main assumption. On the other hand, if $a_{ij} + f(\alpha) > 0$ for all $\alpha \in (0, x_q)$, then $q \notin \{i, j\}$ and the definitions of $f(\alpha)$, B_α , and \mathbf{y}_α make sense for $\alpha = x_q$ as well. Letting $\alpha = x_q$, we obtain $y_q = 0$, contradicting Claim 2.3 and completing the proof of Claim 2.6. \square

Claim 2.7 *The graph \overline{G} is triangle-free.*

Proof Assume the assertion false and let $i, j, k \in [n]$ be such that $ij, ik, jk \in E(\overline{G})$. Let the line given by

$$(C_i - C_k)x + (C_j - C_k)y = 0 \quad (19)$$

intersect the triangle formed by the lines $x = -x_i$, $y = -x_j$, $x + y = x_k$ at some point (α, β) . Let $\mathbf{y} = (x_1 + \Delta_1, \dots, x_n + \Delta_n)$, where

$$\Delta_i = \alpha, \quad \Delta_j = \beta, \quad \Delta_k = -\alpha - \beta, \quad \text{and} \quad \Delta_l = 0 \text{ for } l \in [n] \setminus \{i, j, k\}.$$

Clearly, $\mathbf{y} \in \mathcal{X}(n)$; Taylor's expansion ((10) and equation (19) give

$$L_2(A, \mathbf{y}) - L_2(A, \mathbf{x}) = C_i\alpha + C_j\beta - C_k(\alpha + \beta) = 0;$$

thus $(A, \mathbf{y}) \in \mathcal{S}_n(c)$. Taylor's expansion (11), Lagrange's condition (12), and equation (19) give

$$\begin{aligned} L_r(A, \mathbf{y}) - L_r(A, \mathbf{x}) &= D_i\alpha + D_j\beta - D_k(\alpha + \beta) \\ &= \mu(\alpha + \beta - \alpha - \beta) + \lambda((C_i - C_k)\alpha + (C_j - C_k)\beta) = 0, \end{aligned}$$

contradicting Claim 2.3 as \mathbf{y} has a zero entry. The proof of Claim 2.7 is completed. \square

Using the following claim, we shall prove that \overline{G} is a specific bipartite graph.

Claim 2.8 *Let the vertices i, j, k satisfy $ij \in E(G)$, $ik \in E(\overline{G})$, $jk \in E(\overline{G})$. Then*

$$(C_i - C_k)(C_j - C_k) > 0.$$

Proof Note first that by Claim 2.5 we have $C_i \neq C_k$ and $C_j \neq C_k$. Consider the hyperbola defined by

$$(C_i - C_k)x + (C_j - C_k)y + xy = 0, \quad (20)$$

and write H for its branch containing the origin. Obviously $(C_i - C_k)(C_j - C_k) < 0$ implies that $\alpha\beta > 0$ for all $(\alpha, \beta) \in H$.

Suppose $(\alpha, \beta) \in H$ is sufficiently close to the origin and let $\mathbf{y} = (x_1 + \Delta_1, \dots, x_n + \Delta_n)$, where

$$\Delta_i = \alpha, \quad \Delta_j = \beta, \quad \Delta_k = -\alpha - \beta, \quad \text{and} \quad \Delta_l = 0 \text{ for } l \in [n] \setminus \{i, j, k\}.$$

Clearly, $\mathbf{y} \in \mathcal{X}(n)$; Taylor's expansion (10) and equation (20) give

$$L_2(A, \mathbf{y}) - L_2(A, \mathbf{x}) = C_i\alpha + C_j\beta - C_k(\alpha + \beta) + \alpha\beta = 0;$$

thus $(A, \mathbf{y}) \in \mathcal{S}_n(c)$. Taylor's expansion (11), Lagrange's condition (12), and equation (20) give

$$\begin{aligned} L_r(A, \mathbf{y}) - L_r(A, \mathbf{x}) &= D_i\alpha + D_j\beta - D_k(\alpha + \beta) + D_{ij}\alpha\beta \\ &= \lambda(C_i\alpha + C_j\beta - C_k(\alpha + \beta)) + D_{ij}\alpha\beta = (D_{ij} - \lambda)\alpha\beta. \end{aligned}$$

Since $D_{ij} < \lambda$ and $L_r(A, \mathbf{y}) \leq L_r(A, \mathbf{x})$, we see that $\alpha\beta < 0$. Thus, $(C_i - C_k)(C_j - C_k) > 0$, completing the proof of Claim 2.8. \square

Claim 2.9 \overline{G} is a bipartite graph and its vertex classes U^+ and U^- can be selected so that $C_u > C_w$ for all $u \in U^+$ and $w \in U^-$ such that $uw \in E(\overline{G})$.

Proof Since $C_i \neq C_j$ for every $ij \in E(\overline{G})$, if \overline{G} has an odd cycle, there exist three consecutive vertices i, k, j along the cycle such that $(C_i - C_k)(C_j - C_k) < 0$. Since \overline{G} is triangle-free, $ij \in E(G)$; hence the existence of the vertices i, j, k contradicts Claim 2.8. Thus, \overline{G} is bipartite.

Claim 2.8 implies that for every $u \in [n]$, the value $C_u - C_v$ has the same sign for every v such that $uv \in E(\overline{G})$. Let U^+ be the set of vertices for which this sign is positive, and let $U^- = [n] \setminus U^+$. Clearly, for every $uv \in E(\overline{G})$, if $u \in U^+$, then $v \in U^-$, and if $u \in U^-$, then $v \in U^+$. Hence, U^+ and U^- partition properly the vertices of \overline{G} , completing the proof of Claim 2.8. \square

Hereafter we suppose that the vertex classes U^+ and U^- of \overline{G} are selected to satisfy the condition of Claim 2.9. Note that U^+ and U^- induce complete graphs in G .

Claim 2.10 G contains an induced 4-cycle.

Proof Assume the assertion false. For every vertex u , write $N(u)$ for the set of its neighbors in the vertex class opposite to its own class.

If there exist $u, v \in U^+$ such that $N(u) \setminus N(v) \neq \emptyset$ and $N(v) \setminus N(u) \neq \emptyset$, taking $x \in N(u) \setminus N(v)$ and $y \in N(v) \setminus N(u)$, we see that $\{x, y, u, v\}$ induces a 4-cycle in G ; thus we will assume that $N(u) \subset N(v)$ or $N(v) \subset N(u)$ for every $u, v \in U^+$. This condition implies that there is a vertex $u_1 \in U^+$ such that $N(v) \subset N(u_1)$ for every $v \in U^+$. By symmetry, there is a vertex $u_2 \in U^-$ such that $N(v) \subset N(u_2)$ for every $v \in U^-$.

If $N(u_1) \neq U^-$ and $N(u_2) \neq U^+$, take $x \in U^- \setminus N(u_1)$ and $y \in U^+ \setminus N(u_2)$, and note that $N(x) = \emptyset$ and $N(y) = \emptyset$. Hence, adding the edge xy to $E(G)$, we see that $L_r(A, \mathbf{x})$ remains the same, while $L_2(A, \mathbf{x})$ increases, contradicting that $\varphi_r(n, c)$ is increasing in c (Proposition 2.2). Thus, either $N(u_1) = U^-$ or $N(u_2) = U^+$, so one of the vertices u_1 or u_2 is connected to every vertex other than itself.

By symmetry, suppose that the vertex n is connected to every vertex of G other than itself. Set $\mathbf{y} = (x_1, \dots, x_{n-1})$ and let B be the principal submatrix of A in the first $(n-1)$ columns. Since

$$x_1 + \dots + x_{n-1} = 1 - x_n, \quad (21)$$

$$L_2(B, \mathbf{y}) = c - x_n(1 - x_n), \quad (22)$$

and

$$L_r(A, \mathbf{x}) = x_n L_{r-1}(B, \mathbf{y}) + L_r(B, \mathbf{y}),$$

we see that $x_n L_{r-1}(B, \mathbf{y}) + L_r(B, \mathbf{y})$ is minimum subject to (21) and (22). Since $B \in \mathcal{A}(n-1)$, by the main assumption, both $L_{r-1}(B, \mathbf{z})$ and $L_r(B, \mathbf{z})$ attain a minimum on a complete graph H and for the same vector \mathbf{z} . Since n is joined to every vertex of H , the minimum $\varphi_r(n, c)$ is attained on a complete graph too, a contradiction completing the proof of Claim 2.10. \square

For convenience, an induced 4-cycle in G will be denoted by a quadruple (i, j, k, l) , where i, j, k, l are the vertices of the cycle, arranged so that $i, j \in U^+$, $k, l \in U^-$, $ik \notin E(G)$, and $jl \notin E(G)$.

Claim 2.11 *If (i, j, k, l) is an induced 4-cycle in G , then $D_{ij} + D_{kl} < D_{jk} + D_{li}$.*

Proof Indeed, let L be the line defined by

$$(C_i - C_k)x + (C_j - C_l)y = 0. \quad (23)$$

Since $i, j \in U^+$ and $k, l \in U^-$, we have $C_i > C_k$ and $C_j > C_l$; thus $xy < 0$ for all $(x, y) \in L$. Suppose that $\alpha \in (0, x_k)$, $\beta \in (-x_j, 0)$, and $(\alpha, \beta) \in L$. Let $\mathbf{y}_\alpha = (x_1 + \Delta_1, \dots, x_n + \Delta_n)$, where

$$\Delta_i = \alpha, \quad \Delta_j = \beta, \quad \Delta_k = -\alpha, \quad \Delta_l = -\beta, \quad \text{and} \quad \Delta_h = 0 \text{ for } h \in [n] \setminus \{i, j, k, l\}.$$

Clearly, $\mathbf{y}_\alpha \in \mathcal{X}(n)$; Taylor's expansion (10) and equation (23) give

$$L_2(A, \mathbf{y}_\alpha) - L_2(A, \mathbf{x}) = (C_i - C_k)\alpha + (C_j - C_l)\beta + \alpha\beta - \alpha\beta + \alpha\beta - \alpha\beta = 0;$$

thus $(A, \mathbf{y}_\alpha) \in \mathcal{S}_n(c)$. Taylor's expansion (11), Lagrange's condition (12), and equation (23) give

$$\begin{aligned} L_r(A, \mathbf{y}_\alpha) - L_r(A, \mathbf{x}) &= D_i\alpha + D_j\beta - D_k\alpha - D_l\beta + (D_{ij} - D_{jk} + D_{kl} - D_{li})\alpha\beta \\ &= \lambda(C_i\alpha + C_j\beta - C_k\alpha - C_l\beta) + (D_{ij} - D_{jk} + D_{kl} - D_{li})\alpha\beta \\ &= (D_{ij} - D_{jk} + D_{kl} - D_{li})\alpha\beta. \end{aligned}$$

Since $L_r(A, \mathbf{y}_\alpha) \geq L_r(A, \mathbf{x})$ and $\alpha\beta < 0$, we find that $D_{ij} + D_{kl} \leq D_{jk} + D_{li}$. If $D_{ij} + D_{kl} = D_{jk} + D_{li}$, setting

$$\alpha = \min \left\{ x_k, \frac{C_j - C_l}{C_i - C_k} x_j \right\},$$

we see that $L_r(A, \mathbf{y}_\alpha) = L_r(A, \mathbf{x})$ and either the k th or the j th entry of \mathbf{y}_α is zero, contradicting Claim 2.3. Hence, $D_{ij} + D_{kl} < D_{jk} + D_{li}$, completing the proof of Claim 2.11. \square

Select an induced 4-cycle (i, j, k, l) and let us investigate D_{ij}, D_{kl}, D_{jk} , and D_{li} in the light of Claim 2.11. We have

$$\begin{aligned} D_{ij} &= \sum \{x_{i_1} \cdots x_{i_{r-2}} : \{i, j, i_1, \dots, i_{r-2}\} \text{ induces an } r\text{-clique}\}, \\ D_{kl} &= \sum \{x_{i_1} \cdots x_{i_{r-2}} : \{k, l, i_1, \dots, i_{r-2}\} \text{ induces an } r\text{-clique}\}, \\ D_{jk} &= \sum \{x_{i_1} \cdots x_{i_{r-2}} : \{j, k, i_1, \dots, i_{r-2}\} \text{ induces an } r\text{-clique}\}, \\ D_{li} &= \sum \{x_{i_1} \cdots x_{i_{r-2}} : \{l, i, i_1, \dots, i_{r-2}\} \text{ induces an } r\text{-clique}\}. \end{aligned}$$

First note that if a product $x_{i_1} \cdots x_{i_{r-2}}$ is present in any of the above sums, then $\{i_1, \dots, i_{r-2}\} \cap \{i, j, k, l\} \neq \emptyset$. Also, a product $x_{i_1} \cdots x_{i_{r-2}}$ is present in both D_{ij} and D_{kl} exactly when it is present in both D_{jk} and D_{li} . Hence, Claim 2.11 implies that there exists a set $\{i_1, \dots, i_{r-2}\}$ such that either $\{j, k, i_1, \dots, i_{r-2}\}$ or $\{l, i, i_1, \dots, i_{r-2}\}$ induces an r -clique, but neither $\{i, j, i_1, \dots, i_{r-2}\}$ nor $\{k, l, i_1, \dots, i_{r-2}\}$ induces an r -clique. This is a contradiction for $r = 3$, as either $\{p, i, j\}$ or $\{p, k, l\}$ induces a triangle for every vertex $p \notin \{i, j, k, l\}$.

Let now $r = 4$. We shall reach a contradiction by proving that $D_{ij} + D_{kl} \geq D_{jk} + D_{li}$. Let D_{ij}^* be the sum of all products $x_p x_q$ present in D_{ij} but not present in any of D_{jk}, D_{kl}, D_{il} . Defining the sums D_{jk}^*, D_{kl}^* , and D_{il}^* likewise, we see that

$$D_{ij} + D_{kl} - D_{jk} - D_{li} = D_{ij}^* + D_{kl}^* - D_{jk}^* - D_{li}^*,$$

so it suffices to prove $D_{ij}^* + D_{kl}^* - D_{jk}^* - D_{li}^* \geq 0$. To this end, write $\Gamma(u)$ for the set of neighbors of a vertex u and set

$$\begin{aligned} A &= \Gamma(i) \setminus \Gamma(k), & B &= \Gamma(j) \setminus \Gamma(l), & X &= A \cap B, \\ C &= \Gamma(k) \setminus \Gamma(i), & D &= \Gamma(l) \setminus \Gamma(j), & Y &= C \cap D, \\ a &= \sum_{p \in A} x_p, & b &= \sum_{p \in B} x_p, & c &= \sum_{p \in C} x_p, & d &= \sum_{p \in D} x_p, & x &= \sum_{p \in X} x_p, & y &= \sum_{p \in Y} x_p. \end{aligned}$$

Observe that A, B and X are subsets of $U^+ \setminus \{i, j\}$, while C, D and Y are subsets of $U^- \setminus \{k, l\}$. For reader's sake, here is an alternative view on A, B, C, D, X , and Y :

$$\begin{aligned} A \setminus X &= \Gamma(i) \cap \Gamma(j) \cap \Gamma(l) \setminus \Gamma(k), & B \setminus X &= \Gamma(i) \cap \Gamma(j) \cap \Gamma(k) \setminus \Gamma(l), \\ C \setminus Y &= \Gamma(k) \cap \Gamma(l) \cap \Gamma(j) \setminus \Gamma(i), & D \setminus Y &= \Gamma(k) \cap \Gamma(l) \cap \Gamma(i) \setminus \Gamma(j), \\ X &= \Gamma(i) \cap \Gamma(j) \setminus (\Gamma(k) \cup \Gamma(l)), & Y &= \Gamma(k) \cap \Gamma(l) \setminus (\Gamma(i) \cup \Gamma(j)), \end{aligned}$$

Let the product $x_p x_q$ be present in D_{jk}^* ; thus $\{j, k, p, q\}$ induces an 4-clique, but neither $\{i, j, p, q\}$ nor $\{k, l, p, q\}$ induces an 4-clique. Clearly, p and q belong to different vertex classes of \overline{G} , say $p \in U^+$ and $q \in U^-$. Since i, j , and k are joined to p , we must have $pl \notin E(G)$, and so $p \in B \setminus X$; likewise we find that $q \in C \setminus Y$. Thus

$$D_{jk}^* \leq \sum_{u \in B \setminus X} x_u \sum_{u \in C \setminus Y} x_u = (b - x)(c - y), \quad (24)$$

and by symmetry,

$$D_{il}^* \leq \sum_{u \in A \setminus X} x_u \sum_{u \in D \setminus Y} x_u = (a - x)(d - y). \quad (25)$$

For every pair (p, q) satisfying

$$p \in X, q \in B \setminus X, \quad \text{or} \quad p \in A \setminus X, q \in X, \quad \text{or} \quad p \in A \setminus X, q \in B \setminus X,$$

we see that $\{i, j, p, q\}$ induces an 4-clique, but p is not joined to k and q is not joined to l ; thus $x_p x_q$ is present in D_{ij}^* . Therefore,

$$D_{ij}^* \geq \sum_{u \in X} x_u \sum_{u \in B \setminus X} x_u + \sum_{u \in A \setminus X} x_u \sum_{u \in X} x_u + \sum_{u \in A \setminus X} x_u \sum_{u \in B \setminus X} x_u = ab - x^2, \quad (26)$$

and by symmetry,

$$D_{kl}^* \geq \sum_{u \in Y} x_u \sum_{u \in D \setminus Y} x_u + \sum_{u \in C \setminus Y} x_u \sum_{u \in Y} x_u + \sum_{u \in C \setminus Y} x_u \sum_{u \in D \setminus Y} x_u = cd - y^2. \quad (27)$$

Now adding (26) and (27), and subtracting (24) and (25), we obtain

$$\begin{aligned} D_{ij}^* + D_{kl}^* - D_{jk}^* - D_{li}^* &\geq ab - x^2 + cd - y^2 - (b - x)(c - y) - (a - x)(d - y) \\ &= (a - c)(b - d) + x(c + d) + y(a + b) - (x + y)^2. \end{aligned}$$

Hence, using $x \leq \min(a, b)$, $y \leq \min(c, d)$, and the inequalities

$$\begin{aligned} a - c &= \sum_{u \in \Gamma(i) \setminus \Gamma(k)} x_u + \sum_{u \in \Gamma(i) \cap \Gamma(k)} x_u - \sum_{u \in \Gamma(k) \setminus \Gamma(i)} x_u - \sum_{u \in \Gamma(i) \cap \Gamma(k)} x_u = C_i - C_k > 0, \\ b - d &= \sum_{u \in \Gamma(j) \setminus \Gamma(l)} x_u + \sum_{u \in \Gamma(j) \cap \Gamma(l)} x_u - \sum_{u \in \Gamma(l) \setminus \Gamma(j)} x_u - \sum_{u \in \Gamma(j) \cap \Gamma(l)} x_u = C_j - C_l > 0, \end{aligned}$$

Lemma 2.1 implies that $D_{ij}^* + D_{kl}^* - D_{jk}^* - D_{li}^* \geq 0$, as required.

This finishes the proof that G is a complete graph for $r = 3, 4$.

Proof of $L_r(A, \mathbf{x}) = \varphi_r(c)$

We know now that G is a complete graph. We have to show that $n = \xi(c)$ and $(x_1, \dots, x_n) = (x, \dots, x, y)$, where x and y are given by (5). Our proof is based on the following assertion.

Claim 2.12 *Let $x_3 \geq x_2 \geq x_1 > 0$ be real numbers satisfying*

$$x_1 + x_2 + x_3 = a, \tag{28}$$

$$x_1x_2 + x_2x_3 + x_3x_1 = b, \tag{29}$$

and let $x_1x_2x_3$ be minimum subject to (28) and (29). Then $x_2 = x_3$.

Proof First note that the hypothesis implies that

$$a^2/4 < b \leq a^2/3. \tag{30}$$

Indeed, the second of these inequalities follows from Maclaurin's inequality; assume for a contradiction that the first one fails. Then, selecting a sufficiently small $\varepsilon > 0$ and setting

$$y_1 = \varepsilon, \quad y_2 = \frac{a - \varepsilon - \sqrt{(a + \varepsilon)^2 - 4(b + \varepsilon^2)}}{2}, \quad y_3 = \frac{a - \varepsilon + \sqrt{(a + \varepsilon)^2 - 4(b + \varepsilon^2)}}{2},$$

we see that y_1, y_2, y_3 satisfy (28), (29), and

$$y_1y_2y_3 = \varepsilon(b - a\varepsilon + \varepsilon^2) < \varepsilon b.$$

Thus, $\min x_1x_2x_3$, subject to (28) and (29), cannot be attained for positive x_1, x_2, x_3 , a contradiction, completing the proof of (30).

By Lagrange's method there exist η and θ such that

$$\begin{aligned}x_1x_2 &= \eta + \theta(x_1 + x_2) = \eta + \theta(a - x_3) \\x_1x_3 &= \eta + \theta(x_1 + x_3) = \eta + \theta(a - x_2) \\x_2x_3 &= \eta + \theta(x_2 + x_3) = \eta + \theta(a - x_1).\end{aligned}$$

If $\theta = 0$ we see that $x_1 = x_2 = x_3$, completing the proof. Suppose $\theta \neq 0$ and assume for a contradiction that $x_2 < x_3$. We find that

$$\begin{aligned}x_1(x_3 - x_2) &= \theta(x_3 - x_2), \\x_2(x_3 - x_1) &= \theta(x_3 - x_1),\end{aligned}$$

and so, $x_1 = x_2$. Solving the system (28,29) with $x_1 = x_2$, we obtain

$$x_3 = \frac{a}{3} + \frac{2}{3}\sqrt{a^2 - 3b}, \quad x_1 = x_2 = \frac{a}{3} - \frac{1}{3}\sqrt{a^2 - 3b},$$

implying that

$$x_1x_2x_3 = \left(\frac{a}{3} + \frac{2}{3}\sqrt{a^2 - 3b}\right) \left(\frac{a}{3} - \frac{1}{3}\sqrt{a^2 - 3b}\right)^2. \quad (31)$$

If $b = a^2/3$, we see that $x_1 = x_2 = x_3$, completing the proof, so suppose that $b < a^2/3$. We shall show that $\min x_1x_2x_3$, subject to (28) and (29), is smaller than the right-hand side of (31). Indeed, setting

$$y_1 = \frac{a}{3} - \frac{2}{3}\sqrt{a^2 - 3b}, \quad y_2 = y_3 = \frac{a}{3} + \frac{1}{3}\sqrt{a^2 - 3b},$$

in view of (30), we see that y_1, y_2, y_3 satisfy (28) and (29). After some algebra we obtain

$$y_1y_2y_3 - x_1x_2x_3 = -\frac{4}{27}(a^2 - 3b)^{3/2} < 0.$$

This contradiction completes the proof of Claim 2.12. \square

Claim 2.12 implies that, out of every three entries of \mathbf{x} , the two largest ones are equal; hence all but the smallest entry of \mathbf{x} are equal. Writing y and x for the smallest and largest entries of \mathbf{x} , we see that x and y satisfy

$$\begin{aligned}\binom{n-1}{2}x^2 + nxy &= c, \\(n-1)x + y &= 1, \\y &\leq x,\end{aligned}$$

and so,

$$y = \frac{1}{n} - \sqrt{1 - 2\frac{n}{n-1}c}, \quad x = \frac{1}{n} + \frac{1}{n}\sqrt{1 - 2\frac{n}{n-1}c}.$$

Since the condition $1 - 2nc/(n-1) \geq 0$ gives

$$n \geq \frac{1}{1-2c},$$

and $y > 0$ gives

$$1 - 2c < \frac{1}{n} + \frac{1}{n^2} < \frac{1}{n-1},$$

we find that $n = \xi(c)$, completing the proof of Theorem 1.2. \square

2.1 Proof of Proposition 1.1

Suppose that $\mathcal{S}_n(c)$ is nonempty and that

$$A \in \mathcal{A}(n), \quad \mathbf{x} \geq 0, \quad L_1(A, \mathbf{x}) = 1, \quad \text{and} \quad L_2(A, \mathbf{x}) = c.$$

Then

$$c = \sum_{1 \leq i < j \leq n} a_{ij} x_i x_j \leq \sum_{1 \leq i < j \leq n} x_i x_j = \frac{1}{2} \left(\sum_i x_i \right)^2 - \frac{1}{2} \sum_i x_i^2 \leq \frac{n-1}{2n} < \frac{1}{2},$$

and so, $c < 1/2$ and $n \geq 1/(1-2c)$; thus $n \geq \lceil 1/(1-2c) \rceil$.

On the other hand, if $c < 1/2$ and $n \geq \lceil 1/(1-2c) \rceil$, let $A \in \mathcal{A}(n)$ be the matrix with all off-diagonal entries equal to 1, and let x, y satisfy

$$\begin{aligned} \binom{n-1}{2} x^2 + (n-1)xy &= c, \\ (n-1)x + y &= 1. \end{aligned}$$

Writing \mathbf{x} for the n -vector (x, \dots, x, y) , we see that $L_1(A, \mathbf{x}) = 1$ and $L_2(A, \mathbf{x}) = c$; thus $\mathcal{S}_n(c)$ is nonempty, completing the proof. \square

3 Upper bounds on $k_r(n, m)$

In this section we prove Theorem 1.3. We start with some facts about Turán graphs.

The s -partite Turán graph $T_s(n)$ is a complete s -partite graph on n vertices with each vertex class of size $\lfloor n/s \rfloor$ or $\lceil n/s \rceil$. Setting $t_s(n) = e(T_s(n))$, after some algebra we obtain

$$t_s(n) = \frac{s-1}{2s} n^2 - \frac{t(s-t)}{2s},$$

where t is the remainder of $n \bmod s$; hence,

$$\frac{s-1}{2s} n^2 - \frac{s}{8} \leq t_s(n) \leq \frac{s-1}{2s} n^2. \quad (32)$$

It is known that the second one of these inequalities can be extended for all $2 \leq r \leq s$:

$$k_r(T_s(n)) \leq \binom{s}{r} \left(\frac{n}{s}\right)^r. \quad (33)$$

The Turán graphs play an exceptional role for the function $k_r(n, m)$: indeed, a result of Bollobás [2] implies that if G is a graph with n vertices and $t_s(n)$ edges, then $k_r(G) \geq k_r(T_s(n))$; hence,

Fact 3.1 $k_r(n, t_s(n)) = k_r(T_s(n))$. □

Thus to simplify our presentation, we assume that $n \geq s \geq r \geq 3$ are fixed integers and m is an integer satisfying $t_{s-1}(n) < m \leq t_s(n)$.

First we define a class of graphs giving upper bounds on $k_r(n, m)$.

The graphs $H(n, m)$

We shall construct a graph $H(n, m)$ with n vertices and m edges, where n, s , and m satisfy $n \geq s \geq 3$ and $t_{s-1}(n) < m \leq t_s(n)$. Note that the construction of $H(n, m)$ is independent of r .

First we define a sequence of graphs $H_0, \dots, H_{\lfloor n/s \rfloor}$ satisfying

$$t_{s-1}(n) = e(H_0) < e(H_1) < \dots < e(H_{\lfloor n/s \rfloor}) = t_s(n), \quad (34)$$

and then we construct $H(n, m)$ using $H_0, \dots, H_{\lfloor n/s \rfloor}$.

The graphs $H_0, \dots, H_{\lfloor n/s \rfloor}$

For every $0 \leq i \leq \lfloor n/s \rfloor$, let H_i be the complete s -partite graph with vertex classes I, V_1, \dots, V_{s-1} such that $|I| = i$ and

$$\lfloor (n-i)/(s-1) \rfloor = |V_1| \leq \dots \leq |V_{s-1}| = \lceil (n-i)/(s-1) \rceil.$$

Note that H_0 is the $(s-1)$ -partite Turán graph $T_{s-1}(n)$, but it is convenient to consider it s -partite with an empty vertex class I . Note also that $H_{\lfloor n/s \rfloor} = T_s(n)$.

The transition from H_i to H_{i+1} can be briefly summarized as follows: select V_j with $|V_j| = \lceil (n-i)/(s-1) \rceil$ and move a vertex u from V_j to I .

In particular, we see that

$$e(H_{i+1}) - e(H_i) = \lceil (n-i)/(s-1) \rceil - i > 0,$$

implying in turn (34).

3.1 Constructing $H(n, m)$

Let I, V_1, \dots, V_{s-1} be the vertex classes of H_i . Select V_j with $|V_j| = \lceil (n-i)/(s-1) \rceil$, select a vertex $u \in V_j$, let $l = \lceil (n-i)/(s-1) \rceil - 1$, and suppose that $V_j \setminus \{u\} = \{v_1, \dots, v_l\}$. Do the following steps:

- (a) remove all edges joining u to vertices in I ;

(b) move u from V_j to I , keeping all edges incident to u ;
(c) for $m = e(H_i) + 1, \dots, e(H_{i+1})$ join u to $v_{m-e(H_i)}$ and write $H(n, m)$ for the resulting graph.

Two observations are in place: first, $e(H(n, m)) = m$, and second, $H(n, e(H_i)) = H_i$ for every $i = 1, \dots, \lfloor n/s \rfloor$.

Note also that every additional edge in step (c) increases the number of r -cliques by $k_{r-2}(H')$, where H' is the fixed graph induced by the set $[n] \setminus (I \cup V_j)$. We thus make the following

Claim 3.2 *The function $k_r(H(n, m))$ increases linearly in m for $e(H_{i-1}) \leq m \leq e(H_i)$.*

We need also the following upper bound on $k_r(H_i)$.

Claim 3.3

$$k_r(H_i) \leq \binom{s-1}{r-1} \left(\frac{n-i}{s-1} \right)^{r-1} i + \binom{s-1}{r} \left(\frac{n-i}{s-1} \right)^r$$

Proof Let I, V_1, \dots, V_{s-1} be the vertex classes of H_i . Since the sizes of the sets V_1, \dots, V_{s-1} differ by at most 1, we see that the set $V_1 \cup \dots \cup V_{s-1}$ induces the Turán graph $T_{s-1}(n-i)$. Hence a straightforward counting gives

$$k_r(H_i) \leq k_{r-1}(T_{s-1}(n-i))i + k_r(T_{s-1}(n-i)),$$

and the claim follows from inequality (33). □

3.2 Proof of Theorem 1.3

Assume that x is a real number satisfying

$$\frac{s-2}{2(s-1)}n^2 < x \leq \frac{s-1}{2s}n^2.$$

and define the functions $p = p(x)$ and $q = q(x)$ by

$$p \geq q, \tag{35}$$

$$(s-1)p + q = n, \tag{36}$$

$$\binom{s-1}{2}p^2 + (s-1)pq = x. \tag{37}$$

We note that

$$p(x) = \frac{1}{s} \left(n + \sqrt{n^2 - \frac{2s}{s-1}x} \right), \quad q(x) = \frac{1}{s} \left(n - (s-1) \sqrt{n^2 - \frac{2s}{s-1}x} \right).$$

Set

$$f(x) = \binom{s-1}{r} p^r + \binom{s-1}{r-1} p^{r-1} q, \tag{38}$$

and note that $f(x) = \varphi_r(x/n^2)n^r$; hence, to prove Theorem 1.3, it is enough to show that if

$$\frac{s-2}{2(s-1)}n^2 < m \leq \frac{s-1}{2s}n^2,$$

then

$$k_r(n, m) \leq f(m) + \frac{n^r}{n^2 - 2m}. \quad (39)$$

We first introduce the auxiliary function $\widehat{f}(x)$, defined for $x \in [t_{s-1}(n), t_s(n)]$ by

$$\widehat{f}(x) = \begin{cases} f\left(x + \frac{s-1}{8}\right), & \text{if } t_{s-1}(n) < x \leq \frac{s-1}{2s}n^2 - \frac{s-1}{8}; \\ f\left(\frac{s-1}{2s}n^2\right), & \text{if } \frac{s-1}{2s}n^2 - \frac{s-1}{8} < x \leq t_s(n). \end{cases}$$

To finish the proof of Theorem 1.3 we first show that

$$k_r(H(n, m)) \leq \widehat{f}(m), \quad (40)$$

and then derive (39) using Taylor's expansion and the fact that $k_r(n, m) \leq k_r(H(n, m))$.

Claim 3.4 *If $m = e(H_i)$, then*

$$k_r(H_i) \leq f\left(m - t_{s-1}(n-i) + \frac{s-2}{2(s-1)}(n-i)^2\right).$$

Proof Indeed, as mentioned above, the set $V_1 \cup \dots \cup V_{s-1}$ induces a $T_{s-1}(n-i)$; hence,

$$i(n-i) + t_{s-1}(n-i) = m,$$

and so,

$$i(n-i) + \frac{s-1}{2s}(n-i)^2 = m - t_{s-1}(n-i) + \frac{s-1}{2s}(n-i)^2.$$

Set

$$m' = m - t_{s-1}(n-i) + \frac{s-1}{2s}(n-i)^2$$

and note that $i = q(m')$. In view of Claim 3.3, we obtain

$$k_r(H_i) \leq \binom{s-1}{r-1} \left(\frac{n-i}{s-1}\right)^{r-1} i + \binom{s-1}{r} \left(\frac{n-i}{s-1}\right)^r = f(m'),$$

completing the proof. □

Claim 3.5 $f'(x) = \binom{s-2}{r-2}p^{r-2}$.

Proof From (38) we have

$$f(x) = \binom{s-1}{r-1} \left(\frac{s-r}{r} p^r + p^{r-1} q \right),$$

and so,

$$f'(x) = \binom{s-1}{r-1} ((s-r) p^{r-1} p' + (r-1) p^{r-2} q p' + p^{r-1} q').$$

From (36) and (37) we have

$$(s-1) p' + q' = 0$$

and

$$(s-1) ((s-2) p p' + p' q + p q') = (s-1) p' (q - p) = x' = 1.$$

Now the claim follows after simple algebra. \square

We immediately see that $f(x)$ is increasing. Also, since $p(x)$ is decreasing, $f'(x)$ is decreasing too, implying that $f(x)$ is concave. This, in turn, implies that $\widehat{f}(x)$ is concave.

For every $i = 1, \dots, \lfloor n/s \rfloor$, by Claim 3.4, we have

$$k_r(H_i) \leq f(m') \leq \widehat{f}(m),$$

and since, by Claim 3.2, $k_r(H(n, m))$ is linear for $m \in [e(H_i), e(H_{i+1})]$, inequality (40) follows.

To finish the proof of (39), note that by Taylor's formula, in view of the concavity of $f(x)$, we have

$$\begin{aligned} \widehat{f}(m) &\leq f\left(m + \frac{s-1}{8}\right) \leq f(m) + \frac{s-1}{8} f'(m) = f(m) + \frac{s-1}{8} \binom{s-2}{r-2} p^{r-2} \\ &\leq f(m) + \frac{s-1}{8} \binom{s-2}{r-2} \left(\frac{n}{s-1}\right)^{r-2} < f(m) + s n^{r-2} \leq f(m) + \frac{n^r}{n^2 - 2m}, \end{aligned}$$

completing the proof of Theorem 1.3. \square

Acknowledgement Thanks to Cecil Rousseau for helpful discussions and to Alex Razborov for pointing out some mistakes in the initial version of the manuscript.

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